

Directed percolation process advected by the compressible flow

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ANNOTATION

It will be shown how the directed percolation process in the presence of compressible velocity fluctuations could be formulated within the means of field-theoretic formalism, which is suitable for the renormalization group treatment.

INTRODUCTION

The directed percolation (DP) process [1] is one of the most important model, that describes formation of the fractal structures. The distinctive property of DP is the exhibition of non-equilibrium second order phase transition [2] between absorbing (inactive) and active state. Similar to the equilibrium critical behavior, emerging scale invariant behavior, can be analyzed with the help of renormalization group (RG) technique. The deviations from the ideal models are known to have a profound effect. The main aim of this study is to describe how the directed percolation process in the presence of compressible velocity fluctuations can be analyzed in the framework of field-theoretic formulation.

THE MODEL

The continuum description of DP in terms of a density field $\psi = \psi(t, \mathbf{x})$ arises from a coarse-graining procedure in which a large number of microscopic degrees of freedom were averaged out. The mathematical model has to respect the absorbing state condition, that is $\psi = 0$ is always a stationary state. The coarse grained stochastic equation then reads [3]

$$\partial_t \psi = D_0(\nabla^2 - \tau_0)\psi - \frac{g_0 D_0}{2} \psi^2 + \eta, \quad (1)$$

where η denotes the noise term, $\partial_t = \partial/\partial t$ is the time derivative, ∇^2 is the Laplace operator, D_0 is the diffusion constant, g_0 is the coupling constant and τ_0 measures deviation

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from the criticality. The Gaussian noise term η with zero mean stands for the neglected fast microscopic degrees of freedom. Its correlation function must respect absorbing state condition and it can be chosen in the following form

$$\langle \eta(t_1, \mathbf{x}_1) \eta(t_2, \mathbf{x}_2) \rangle = g_0 D_0 \psi(t_1, \mathbf{x}_1) \delta(t_1 - t_2) \delta^{(d)}(\mathbf{x}_1 - \mathbf{x}_2). \quad (2)$$

The next step consists in the incorporation of the velocity fluctuations into the equation (1). The standard route based on the replacement ∂_t by the Lagrangian derivative $\partial_t + (\mathbf{v} \cdot \nabla)$ is not sufficient due to the assumed compressibility. As was shown in [4] the additional parameter a_0 has to be introduced via following replacement

$$\partial_t \rightarrow \partial_t + (\mathbf{v} \cdot \nabla) + a_0 (\nabla \cdot \mathbf{v}). \quad (3)$$

The choice $a_0 = 1$ corresponds to the conserved quantity ψ , whereas for the choice $a_0 = 0$ the conserved quantity is $\tilde{\psi}$. The full description of the model requires specification of the velocity field. Following the work [5] the velocity field is considered to be a random Gaussian variable with zero mean and correlator

$$\langle v_i(t, \mathbf{x}) v_j(0, \mathbf{0}) \rangle = \int \frac{d\omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} D_v(\omega, \mathbf{k}) e^{-i\omega t + \mathbf{k} \cdot \mathbf{x}}, \quad (4)$$

where d is dimension of the space and the kernel function $D_v(\omega, \mathbf{k})$ is chosen in the form

$$D_v(\omega, \mathbf{k}) = [P_{ij}^k + \alpha Q_{ij}^k] \frac{g_{10} u_{10} D_0^3 k^{4-d-y-\eta}}{\omega^2 + u_{10}^2 D_0^2 (k^{2-\eta})^2}. \quad (5)$$

Here $P_{ij}^k = \delta_{ij} - k_i k_j / k^2$ is transverse and Q_{ij}^k longitudinal projection operator, $k = |\mathbf{k}|$, positive parameter $\alpha > 0$ can be interpreted as a deviation from the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. The coupling constant g_{10} and exponent y describe the equal-time velocity correlator or equivalently, the energy spectrum of the velocity fluctuations. On the other hand parameter $u_{10} > 0$ and exponent η describe dispersion behavior of the mode k .

The exponents y and η are analogous to the standard expansion parameter $\varepsilon = 4 - d$ in the static critical phenomena [6]. According to the general rules of the RG approach we formally assume that the exponents ε, y and η are of the same order of magnitude and in principle they constitute small expansion parameters in a perturbation sense.

For the effective use of RG method it is advantageous to reformulate the stochastic problem (1-5) into the field-theoretic language. This can be achieved in the standard

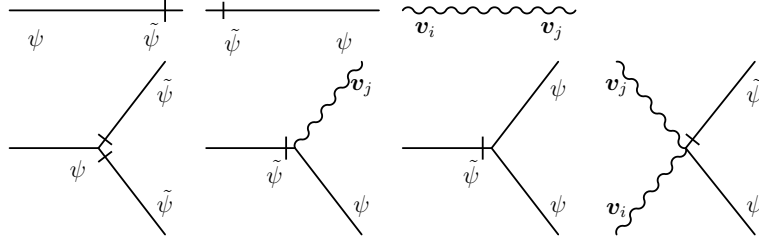


Figure 1. Elements of the perturbation theory in the graphical representation.

fashion [7, 8] and the resulting dynamic functional can be written as a sum

$$\mathcal{J}[\varphi] = \mathcal{J}_{\text{diff}}[\varphi] + \mathcal{J}_{\text{vel}}[\varphi] + \mathcal{J}_{\text{int}}[\varphi], \quad (6)$$

where $\varphi = \{\tilde{\psi}, \psi, \mathbf{v}\}$ stands for the complete set of fields and ψ^\dagger is the response field. The corresponding terms have the following form

$$\mathcal{J}_{\text{diff}}[\varphi] = \int dt \int d^d \mathbf{x} \left\{ \tilde{\psi} [\partial_t - D_0 \nabla^2 + D_0 \tau_0] \psi \right\}, \quad (7)$$

$$\mathcal{J}_{\text{vel}}[\mathbf{v}] = -\frac{1}{2} \int dt_1 \int dt_2 \int d^d \mathbf{x}_1 \int d^d \mathbf{x}_2 \mathbf{v}_i(t_1, \mathbf{x}_1) D_{ij}^{-1}(t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2) \mathbf{v}_j(t_2, \mathbf{x}_2), \quad (8)$$

$$\mathcal{J}_{\text{int}}[\varphi] = \int dt \int d^d \mathbf{x} \tilde{\psi} \left\{ \frac{D_0 \lambda_0}{2} [\psi - \tilde{\psi}] - \frac{u_{20}}{2D_0} \mathbf{v}^2 + (\mathbf{v} \cdot \nabla) + a_0 (\nabla \cdot \mathbf{v}) \right\} \psi. \quad (9)$$

All but third term in (9) stems directly from the nonlinear terms in (1) and (3). The third term proportional to $\propto \tilde{\psi} \psi \mathbf{v}^2$ deserves a special consideration. Presence of such term is prohibited in the original Kraichnan model due to the underlying Galilean invariance. However in our case the finite time correlations of the velocity fluctuations does not impose such restriction. In the language of Feynman graphs, one can show that such term will indeed be generated as can be readily seen considering first three graphs in the following expansion

$$\Gamma_{\tilde{\psi} \psi \mathbf{v} \mathbf{v}} = \frac{u_2}{D} \delta_{ij} Z_8 + \text{[diagrams]} + \frac{1}{2} \text{[diagram]} + \text{[diagram]} + \text{[diagram]} + \text{[diagram]} + \text{[diagram]} + \text{[diagram]} + \text{[diagram]}. \quad (10)$$

We conclude that compressibility and non-Galilean nature of the velocity correlator lead to the quite involved situation, which requires analysis. Note, that in the incompressible case [9] presence of a given term does not lead to the significant effects.

RENORMALIZATION GROUP ANALYSIS

The field-theoretic formulation summarized in (8)-(9) has an advantage to be amenable to the machinery of field theory [6]. Near criticality $\tau = 0$ large fluctuations on all scales dominate the behavior of the system, which results into the divergences in Feynman graphs. The RG technique allows us to deal with them and as a result it allows for perturbative computation of critical exponent in a formal expansion around upper critical dimension. Thus provides us with information about the scaling behavior of Green functions. The renormalization of the model can be achieved through the relations

$$\begin{aligned}
 D_0 &= DZ_D, & \tau_0 &= \tau Z_\tau + \tau_C, & a_0 &= aZ_a, & g_{10} &= g_1 \mu^{y+\eta} Z_{g_1}, \\
 u_{10} &= u_1 \mu^\eta Z_{u_1}, & \lambda_0 &= \lambda \mu^\varepsilon Z_\lambda, & u_{20} &= u_2 Z_{u_2}, \\
 \tilde{\psi} &= Z_{\tilde{\psi}} \tilde{\psi}_R, & \psi &= Z_\psi \psi_R, & \mathbf{v} &= Z_v \mathbf{v}_R.
 \end{aligned} \tag{11}$$

where μ is the reference mass scale in the MS scheme [6].

CONCLUSIONS

In this brief article we have summarized main points of the field-theoretic study of directed percolation process in the presence of compressible velocity field. The detailed results for the renormalization constants and analysis of the scaling behavior will be published elsewhere [10].

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